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Transverse vibrations of an axially accelerating viscoelastic string with geometric nonlinearity

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Abstract. Two-to-one parametric resonance in transverse vibration of an axially accelerating viscoelastic string with geometric nonlinearity is investigated. The transport speed is assumed to be a constant mean speed with small harmonic variations. The nonlinear partial differential equation that governs transverse vibration of the string is derived from Newton's second law. The method of multiple scales is applied directly to the equation, and the solvability condition of eliminating secular terms is established. Closed-form solutions for the amplitude of the vibration and the existence conditions of nontrivial steady-state response in two-to-one parametric resonance are obtained. Some numerical examples showing effects of the mean transport speed, the amplitude and the frequency of speed variation are presented. Lyapunov's linearized stability theory is employed to analyze the stability of the trivial and nontrivial solutions for two-to-one parametric resonance. Some numerical examples highlighting the effects of the related parameters on the stability conditions are presented.

Key words: axially accelerating string, geometric nonlinearity, method of multiple scales, stability, viscoelasticity

1. Introduction

Many engineering devices involve transverse vibrations of axially moving strings. Serpentine belts, fiber windings, magnetic tapes and thread lines all belong to this class. One important problem is the occurrence of large transverse vibrations in such systems. Much research has been done on the aspect, which has been reviewed in [1–4]. To model dissipative mechanisms, viscoelasticity of string material is recently taken into consideration. Fung *et al.* [5] developed a numerical method based on Galerkin's method and finite difference integration for transient response of a moving viscoelastic string that is integrally constituted by the Boltzmann superposition principle. Zhang and Zu applied the method of multiple scales directly to the partial-differential equation that governs the transverse vibration of a moving viscoelastic string, and obtained the nonlinear natural frequencies and the amplitude of the free vibration [6] and the amplitude of near- and exact-resonant steady-state response of the forced vibration [7]. They also obtained closed-form solutions for the amplitude of the vibration and the existence conditions of the nontrivial steady-state response of summation resonance of parametrically excited moving viscoelastic strings [8], and investigated the stability of the trivial and nontrivial responses based on the Lyapunov linearized stability theory [9]. In their work, a linear differential viscoelastic constitutive law, the Kelvin viscoelastic model, is chosen to describe the viscoelastic property of the string material. Chen, Zu and Wu [10] applied the method of multiple scales to study analytically parametric vibration of the axially moving viscoelastic string constituted by the Boltzmann superposition principle. For transverse vibration of a moving viscoelastic string constituted by the general linear differential relation, Zhao and Chen treated the stress as a new variable in the finite-difference scheme, and proposed a numerical algorithm by discreting the partial-differential equation into a large differential-algebraic equation system [11]. However, all the above-mentioned analytical studies [6–10] addressed the constant axial-transport-speed problem. In fact, many real systems can be better represented by the model that the axial transport speed is a constant mean velocity with small periodic fluctuations. The speed fluctuations also lead to tension fluctuations, since the tension is velocity-dependent. Hence the investigation of an axially accelerating string is closely related to comprehensive studies of a constant-speed string with tension fluctuations [8–13]. Although the equation of transverse vibration for an axially accelerating string was first derived by Miranker in 1960 [14], detailed research has not been conducted until recently. Pakdemirili and Batan [15] used the Galerkin method and Floquet theory to treat numerically the dynamic stability of a constantly accelerating string. Pakdemirili *et al.* [16] further considered the dynamic stability of a moving string that the time-dependent axial velocity is sinusoidal. In the case for which the time-dependent axial velocity varies harmonically about a constant mean velocity, Pakdemirli and Ulsoy [17] applied the discretization-perturbation method and the direct-perturbation method to analyze the stability of an axially accelerating string. For the arbitrary time-dependent axial velocity, Ozkaya and Pakdemirli [18] used Lie-group theory to find exact solutions. However, almost all previous research on the transverse vibration of an axially accelerating string is confined to linear models. The literature that is specially related to nonlinear vibrations of axially accelerating strings is very limited. Wu and Chen [19] studied steady-state responses of an axially accelerating elastic string with geometric nonlinearity, but they did not consider the damping effect resulted from the viscoelasticity of the string. To address the lack of research in this aspect, the authors investigated transverse vibration of an axially accelerating viscoelastic string with geometric nonlinearity.

In this paper, the partial-differential equation that governs nonlinear vibrations of an axially accelerating string is established. The method of multiple scales is applied directly to the equation to analyze two-to-one parametric resonance. Solvability conditions, which lead to the differential equations satisfied by the amplitudes and phase angles of the dynamic response, are derived. Closed-form solutions for the amplitude of nontrivial response of two-to-one parametric resonance and the corresponding existence conditions are obtained. Numerical results of steady-state responses and existence boundaries are presented. The Lyapunov linearized stability theory is employed to obtain instability condition of the trivial solution and the stability of the first or second nontrivial solution. Numerical examples show the effects of the mean transport speed, the amplitude and the frequency of speed variation on these conditions.

2. Equation of transverse vibration

Consider a uniform, flexible, axially moving viscoelastic string of density ρ, cross-sectional area A, initial tension P, and uniform transport speed v that travels between two fixed eyelets separated by distance l. The speed v is not constant, but a prescribed function of time t. Several simplifying assumptions are made as follows: (1) only transverse motion in the y-direction is taken into consideration; (2) Lagrangian strain is employed as a finite measure of the strain of the string; (3) the viscoelastic string is in a state of uniform initial stress, and the initial tension is rather large; (4) only geometric nonlinearity due to finite stretching is considered through Lagrangian strain.

Based on the above assumptions, the equation of motion in the transverse direction can be derived from Newton's second law [5,11]

$$\rho A \frac{\mathrm{D}^2 U}{\mathrm{D}t^2} = P \frac{\partial^2 U}{\partial x^2} + \frac{\partial}{\partial x}\left(A\sigma\left(x,t\right)\frac{\partial U\left(x,t\right)}{\partial x}\right), \tag{1}$$

where $U(x, t)$ is the displacement in the transverse direction, x is the spatial Cartesian coordinate in the axial direction and $\sigma(x, t)$ is the perturbed stress.

For axially accelerating strings, the transverse acceleration is compounded by the relative acceleration, the Coriolis acceleration, and the convected acceleration [13]

$$\frac{\mathrm{D}^2 U}{\mathrm{D}t^2} = \frac{\partial^2 U}{\partial t^2} + 2v\frac{\partial^2 U}{\partial x \partial t} + \frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial U}{\partial x} + v^2\frac{\partial^2 U}{\partial x^2}. \tag{2}$$

The Kelvin viscoelastic model is chosen to describe the viscoelastic property of the string material. Thus the stress-strain relation is

$$\sigma\left(x,t\right) = E_0 \varepsilon_L\left(x,t\right) + \eta\frac{\partial \varepsilon_L\left(x,t\right)}{\partial t}, \tag{3}$$

where $\sigma(x, t)$ is the perturbed stress in the axial direction, $\varepsilon_L(x, t)$ is the perturbed Lagrangian strain component, E_0 is the stiffness constant of the string, and η is the dynamic viscosity. For strings with finite amplitude, the perturbed Lagrangian strain component in the axial direction related to the transverse displacement is given by

$$\varepsilon_L\left(x,t\right) = \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2. \tag{4}$$

It is assumed that the transport speed is characterized as a small simple harmonic variation about the constant mean speed, *i.e.*

$$v\left(t\right) = c_0 + c_1\cos\Omega t \qquad \left(c_0, c_1 > 0\right). \tag{5}$$

Substituting Equations (2)–(5) in Equation (1) and transforming the resulting equation into dimensionless form yields

$$\begin{aligned}&\frac{\partial^2 u}{\partial \tau^2} + 2\left(\gamma + \gamma_1\cos\omega\tau\right)\frac{\partial^2 u}{\partial \xi \partial \tau} + \left(\gamma^2 + \frac{\gamma_1^2}{2} + 2\gamma\gamma_1\cos\omega\tau + \frac{\gamma_1^2}{2}\cos 2\omega\tau - 1\right)\frac{\partial^2 u}{\partial \xi^2}\\ &-\omega\gamma_1\sin\omega\tau\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\varepsilon\varsigma\left(\xi,\tau\right)\frac{\partial u(\xi,\tau)}{\partial \xi}\right),\end{aligned} \tag{6}$$

where

$$\varsigma\left(\xi,\tau\right) = \frac{E_e}{2}\left(\frac{\partial v\left(\xi,\tau\right)}{\partial \xi}\right)^2 + \frac{E_v}{2}\frac{\partial}{\partial \tau}\left(\frac{\partial v\left(\xi,\tau\right)}{\partial \xi}\right)^2, \tag{7}$$

$$\begin{aligned}&u = \frac{U}{l},\ \xi = \frac{x}{l},\ \tau = \frac{t}{l}\sqrt{\frac{P}{\rho A}},\ \omega = \Omega l\sqrt{\frac{\rho A}{P}},\ E_e = \frac{E_0 A}{P},\\ &E_v = \frac{\eta b}{l}\sqrt{\frac{P}{\rho A}},\ \gamma = c_0\sqrt{\frac{\rho A}{P}},\ \gamma_1 = c_1\sqrt{\frac{\rho A}{P}},\ \varepsilon\varsigma\left(\xi,\tau\right) = \frac{A\sigma(x,t)}{P},\end{aligned} \tag{8}$$

In Equation (6), a small dimensionless parameter ε is employed as a bookkeeping device, which implies that the disturbed internal force $A\sigma(x, \quad t)$ is much smaller than the initial tension. Equation (6) is the governing equation of transverse motion in dimensionless form. The corresponding non-vibration boundary conditions are

$$u(0,\ t) = 0,\ u(1,\ t) = 0. \tag{9}$$

3. Application of the method of multiple scales

Introduce the mass, gyroscopic, and linear stiffness operators as follows

$$M = I\ ,\ G = 2\gamma\frac{\partial}{\partial\xi}\ ,\ K = \left(\gamma^2 - 1\right)\frac{\partial^2}{\partial\xi^2} \tag{10}$$

where the operators M and K are symmetric and positive definite for subcritical transport speed, G is skew-symmetric and represents a convective Coriolis acceleration component. Since the variation of the transport speed is small, one may let $c_1/c_0 = \gamma_1/\gamma = \varepsilon\delta$. Then substituting Equations (7) and (10) in Equation (6), one obtains a continuous gyroscopic system with some small nonlinear terms and some small-parameter excitation terms, namely

$$\begin{aligned} M\frac{\partial^2 u}{\partial\tau^2} + G\frac{\partial u}{\partial\tau} + Ku = \varepsilon&\left[\frac{3}{2}E_e\left(\frac{\partial u}{\partial\xi}\right)^2\frac{\partial^2 u}{\partial\xi^2} + 2E_v\frac{\partial u}{\partial\xi}\frac{\partial^2 u}{\partial\xi\partial\tau}\frac{\partial^2 u}{\partial\xi^2}\right. \\ &+ E_v\left(\frac{\partial u}{\partial\xi}\right)^2\frac{\partial^3 u}{\partial\xi^2\partial\tau} - \frac{\delta\gamma^2}{2}\left(\delta\varepsilon + 4\cos\omega\tau + \delta\varepsilon\cos 2\omega\tau\right)\frac{\partial^2 u}{\partial\xi^2} \\ &\left.- 2\delta\gamma\cos\omega\tau\frac{\partial^2 u}{\partial\tau\partial\xi} + \omega\delta\gamma\sin\omega\tau\frac{\partial u}{\partial\xi}\right]. \end{aligned} \tag{11}$$

The method of multiple scales will be directly employed to solve Equation (11). A first-order uniform approximation is sought in the form

$$u(\xi, \tau, \varepsilon) = u_0(\xi, T_0, T_1) + \varepsilon u_1(\xi, T_0, T_1) + O\left(\varepsilon^2\right) \tag{12}$$

where $T_0 = \tau$ is a fast scale characterizing motions occurring at ω or ω_m (one of the natural frequencies of the corresponding unperturbed linear continuous gyroscopic system), and $T_1 = \varepsilon\tau$ is a slow scale characterizing the modulation of the amplitudes and phases due to the nonlinearity and possible resonance. Substituting Equation (12) in Equation (11), using the chain rule of time derivatives, and equating coefficients of like powers of ε, one has

$$M\frac{\partial^2 u_0}{\partial T_0^2} + G\frac{\partial u_0}{\partial T_0} + Ku_0 = 0\ , \tag{13}$$

$$\begin{aligned} M\frac{\partial^2 u_1}{\partial T_0^2} + G\frac{\partial u_1}{\partial T_0} + Ku_1 = &-2M\frac{\partial^2 u_0}{\partial T_0\partial T_1} - G\frac{\partial u_0}{\partial T_1} + \frac{3}{2}E_e\left(\frac{\partial u_0}{\partial\xi}\right)^2\frac{\partial^2 u_0}{\partial\xi^2} \\ &+ 2E_v\frac{\partial u_0}{\partial\xi}\frac{\partial^2 u_0}{\partial\xi\partial\tau}\frac{\partial^2 u_0}{\partial\xi^2} + E_v\left(\frac{\partial u_0}{\partial\xi}\right)^2\frac{\partial^3 u_0}{\partial\xi^2\partial\tau} - 2\delta\gamma\cos\omega T_0\frac{\partial^2 u_0}{\partial T_0\partial\xi} \\ &- 2\delta\gamma^2\cos\omega T_0\frac{\partial^2 u_0}{\partial\xi^2} + \omega\delta\gamma\sin\omega T_0\frac{\partial u_0}{\partial\xi}\ , \end{aligned} \tag{14}$$

at leading and first order of ε, respectively.

Under the boundary condition (9), Wickert and Mote [20] gave the solution to Equation (13) as

$$u_0\left(\xi, T_0, T_1\right) = \sum_{m=0,\pm1,\cdots} \left[\phi_m(\xi) A_m(T_1) \mathrm{e}^{\mathrm{i}\omega_m T_0} + \bar{\phi}_k(\xi) \bar{A}_m(T_1) e^{-i\omega_m T_0}\right], \tag{15}$$

where the overbar denotes complex conjugation, and the mth natural frequency and the mth normalized complex eigenfunction of the displacement field are, respectively, given by

$$\omega_m = m\pi\left(1-\gamma^2\right), \ \phi_m\left(\xi\right) = \sqrt{2}\sin\left(m\pi\xi\right)\mathrm{e}^{\mathrm{i}m\pi\gamma\xi}, \tag{16}$$

which satisfy the orthonormality relations

$$\langle \phi_m, M\phi_m \rangle = 1\,, \ \langle \phi_m, G\phi_m \rangle = 2\mathrm{i}m\pi\gamma^2. \tag{17}$$

Here the inner product is the same as that defined in [20], namely

$$\left\langle \phi_i, \phi_j \right\rangle = \int_0^1 \bar{\phi}_i(\xi)\phi_j(\xi)\mathrm{d}\xi. \tag{18}$$

If the variation frequency ω approaches twice any natural frequency of Equation (14), two-to-one parametric resonance may occur. A detuning parameter μ is introduced to quantify the deviation of ω from $2\omega_m$, and ω is described by

$$\omega = 2\omega_m + \varepsilon\mu. \tag{19}$$

To investigate the two-to-one parametric response, Equation (15) can be expressed as

$$u_0\left(\xi, T_0, T_1\right) = \phi_m(\xi) A_m(T_1) \mathrm{e}^{\mathrm{i}\omega_m T_0} + cc, \tag{20}$$

where cc represents complex conjugate of all preceding terms on the right-hand of an equation. Substituting Equations (19) and (20) in Equation (14) yields

$$\begin{aligned} M\tfrac{\partial^2 u_1}{\partial T_0^2} + G\tfrac{\partial u_1}{\partial T_0} + Ku_1 = &\left\{-2\mathrm{i}\omega_m A_m' M\phi_m - A_m' G\phi_m + M_{2m}\left(3E_e + 2\mathrm{i}\omega_m E_v\right) A_m \left|A_m\right|^2 \right. \\ &\left. + \left\{\left[\left(1+\mathrm{i}\right)\omega_m + \tfrac{\mu}{2}\right]\bar{\phi}'_m - \gamma\bar{\phi}''_m\right\}\delta\gamma\bar{A}_m \mathrm{e}^{\mathrm{i}\mu t_1}\right\}\mathrm{e}^{\mathrm{i}\omega_m T_0} + NST + cc\ , \end{aligned} \tag{21}$$

where NST denotes the terms that will not bring secular terms into the solution, and

$$M_{2m} = \frac{1}{2}\left(\frac{\partial\phi_m}{\partial\xi}\right)^2 \frac{\partial^2\bar{\phi}_m}{\partial\xi^2} + \frac{\partial\phi_m}{\partial\xi}\frac{\partial\bar{\phi}_m}{\partial\xi}\frac{\partial^2\phi_m}{\partial\xi^2}. \tag{22}$$

Equation (21) has a bounded solution only if a solvability condition is satisfied. The solvability condition demands that the right side of Equation (21) be orthogonal to every solution of the homogeneous problem. In order to avoid the unbounded solution, the solvability conditions can be described as following

$$\begin{aligned} &-2\mathrm{i}\omega_m A_m' \left\langle M\phi_m, \phi_m\right\rangle - A_m' \left\langle G\phi_m, \phi_m\right\rangle + A_m \left|A_m\right|^2 \left(3E_e + 2\mathrm{i}\omega_m E_v\right)\left\langle M_{2m}, \phi_m\right\rangle \\ &+ \left\{\left[\left(1+\mathrm{i}\right)\omega_m + \tfrac{\mu}{2}\right]\left\langle\bar{\phi}'_m, \phi_m\right\rangle - \gamma\left\langle\bar{\phi}''_m, \phi_m\right\rangle\right\}\delta\gamma\bar{A}_m \mathrm{e}^{\mathrm{i}\mu t_1} = 0 \end{aligned} \tag{23}$$

Using the natural frequencies and the eigenfunctions given by Equation (15), one calculates the inner product in Equation (23), and obtains

$$\langle M_{2m}, \phi_m \rangle = -\frac{1}{4}\pi^4 m^4 \left(3 + 2\gamma^2 + 3\gamma^4\right) , \quad \left\langle \bar{\phi}'_m, \phi_m \right\rangle = 0 ,$$
$$\left\langle \phi''_m, \phi_m \right\rangle = -m^2\pi^2 \left(1 + \gamma^2\right) , \quad \left\langle \bar{\phi}''_m, \phi_m \right\rangle = -\frac{\mathrm{i}m\pi}{2\gamma} \left(1 - \mathrm{e}^{-2i\gamma m\pi}\right) . \tag{24}$$

Substituting Equations (16) and (24) in Equation (23) results in the solvability condition

$$A'_m - \frac{\mathrm{i}}{8}\pi^3 m^3 \left(3 + 2\gamma^2 + 3\gamma^4\right) \left[3E_e + 2\mathrm{i}m \left(1 - \gamma^2\right) E_v\right] A_m \left|A_m\right|^2$$
$$-\frac{\delta\gamma}{2} \sin\left(\gamma m\pi\right) \left[\sin\left(\gamma m\pi\right) + \mathrm{i}\cos\left(\gamma m\pi\right)\right] \bar{A}_m \mathrm{e}^{\mathrm{i}\mu t_1} = 0 . \tag{25}$$

Obviously, Equation (25) has a trivial solution $A_m = 0$. Expressing the nontrivial solution A_m in polar form gives

$$A_m = \alpha_m \mathrm{e}^{\mathrm{i}\beta_m} \tag{26}$$

where α_m and β_m represent respectively the amplitude and the phase angle of the response. Substituting Equation (26) in Equation (25) and separating the resulting equation into real and imaginary parts, one obtains

$$\alpha'_m = -\frac{1}{4}\pi^3 m^4 \left(3 + 2\gamma^2 + 3\gamma^4\right) \left(1 - \gamma^2\right) E_v \alpha_m^3$$
$$+ \frac{\delta\gamma\alpha_m}{4} \left[\cos\theta_m - \cos\left(2\gamma m\pi - \theta_m\right)\right] , \tag{27}$$

$$\theta'_m = \mu - \frac{3}{8}\pi^3 m^3 \left(3 + 2\gamma^2 + 3\gamma^4\right) E_e \alpha_m^2 - \frac{\delta\gamma}{4} \left[\sin\theta_m + \sin\left(2\gamma m\pi - \theta_m\right)\right] , \tag{28}$$

where

$$\theta_m = \mu T_1 - 2\beta_m . \tag{29}$$

4. The amplitude the vibration and the existence conditions of steady-state responses

For the steady-state response, the amplitude α_m and the new phase angle θ_m in Equations (27) and (28) are constant. Setting $\alpha'_m = 0$ and $\theta'_m = 0$in Equations (27) and (28) respectively, and then eliminating θ_m from resulting equations lead to

$$\frac{\pi^6}{16} m^6 \left(3 + 2\gamma^2 + 3\gamma^4\right)^2 \left[4m^2 \left(1 - \gamma^2\right)^2 E_v^2 + 9E_e^2\right] \alpha_m^4$$
$$- \frac{3}{2}\pi^3 \left(3 + 2\gamma^2 + 3\gamma^4\right) \mu E_e m^3 \alpha_m^2 + \mu^2 - \delta^2\gamma^2 \sin^2\left(\gamma m\pi\right) = 0 . \tag{30}$$

It is obvious that Equation (27) possesses a singular point at the origin (trivial zero solution). In addition, there may exist a nontrivial periodic solution with amplitudes, which is defined by Equation (30), namely

$$\alpha_m^2 = \frac{3\mu E_e \pm \sqrt{\delta^2\gamma^2 \left[4m^2 \left(1 - \gamma^2\right)^2 E_v^2 + 9E_e^2\right] \sin^2\left(\gamma m\pi\right) - 4m^2 \left(1 - \gamma^2\right)^2 E_v^2 \mu^2}}{\frac{1}{4}\pi^3 m^3 \left(3 + 2\gamma^2 + 3\gamma^4\right) \left[4m^2 \left(1 - \gamma^2\right)^2 E_v^2 + 9E_e^2\right]} . \tag{31}$$

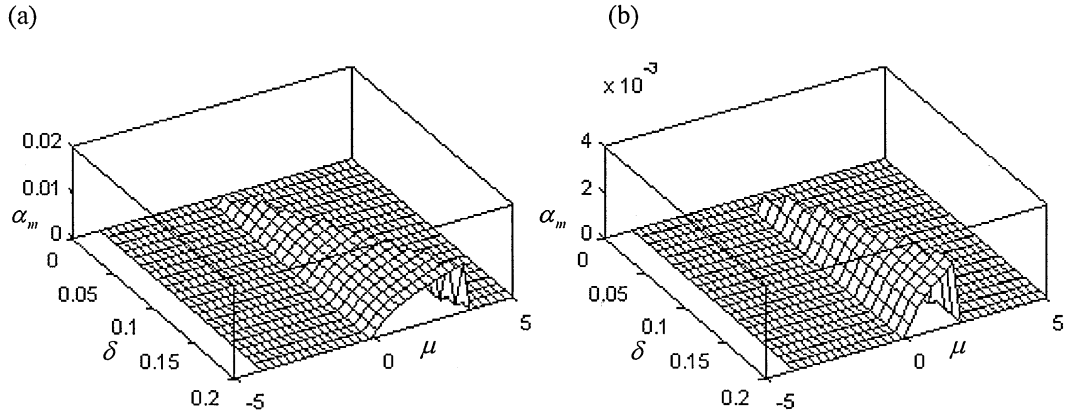

Figure 1. The response amplitude *vs.* δ and μ for fixed $\gamma = 0{\cdot}3$. (a) The first nontrivial solution. (b) The second nontrivial solution.

Equation (31) represents the amplitudes of the steady-state response of the two-to-one parametric resonance.

From equation (31), it can be concluded that the nontrivial steady-state solutions exist only when the following conditions are held,

$$\begin{aligned} &3\mu E_e \pm \sqrt{\delta^2\gamma^2\left[4m^2\left(1-\gamma^2\right)^2 E_v^2 + 9E_e^2\right]\sin^2(\gamma m\pi) - 4m^2\left(1-\gamma^2\right)^2 E_v^2\mu^2} > 0 \\ &\delta^2\gamma^2\left[4m^2\left(1-\gamma^2\right)^2 E_v^2 + 9E_e^2\right]\sin^2(\gamma m\pi) - 4m^2\left(1-\gamma^2\right)^2 E_v^2\mu^2 \geq 0\ . \end{aligned} \tag{32}$$

For a viscoelastic string, $E_V \neq 0$. From Equation (32), the existence condition of nontrivial steady-state solutions can be expressed as

$$\mp\delta\gamma\left|\sin(\gamma m\pi)\right| < \mu \leq \delta\gamma\left|\sin(\gamma m\pi)\right|\sqrt{1+\frac{9E_e^2}{4m^2\left(1-\gamma^2\right)^2 E_v^2}}. \tag{33}$$

Using Equation (31), one can determine the effects of the mean transport speed, the amplitude and the frequency of the speed variation on the amplitudes of the steady-state response. In all the following numerical examples, the authors set $m = 1$, $E_e = 400$ and $E_v = 10$.

The amplitudes of steady-state responses and existence boundaries *vs.* the nondimensional amplitude of speed variation δ and the detuning parameter μ at the fixed nondimensional mean transport speed $\gamma = 0{\cdot}3$ or $\gamma = 0{\cdot}9$ respectively are shown in Figures 1 and 2. The response amplitude increases with the growth of δ and μ. The nondimensional mean transport speed significantly influences the existence boundaries.

The amplitudes of steady-state responses and existence boundaries *vs.* the nondimensional the mean transport speed γ and the detuning parameter μ at the fixed nondimensional amplitude of speed variation $\delta = 0{\cdot}1$ is shown in Figure 3. Mean translation speeds influence not only the amplitude of the nontrivial steady-state response but also their existence region. The response amplitude increases with the growth of μ. The nondimensional amplitude of speed variation significantly influences the existence boundaries.

In all above cases, it is evident that μ has a significant effect on the amplitude and the boundary of existence of the steady-state response. At exact two-to-one parametric resonance ($\mu = 0$), from Equation (33), only the first nontrivial solution exists. The amplitude of steady-

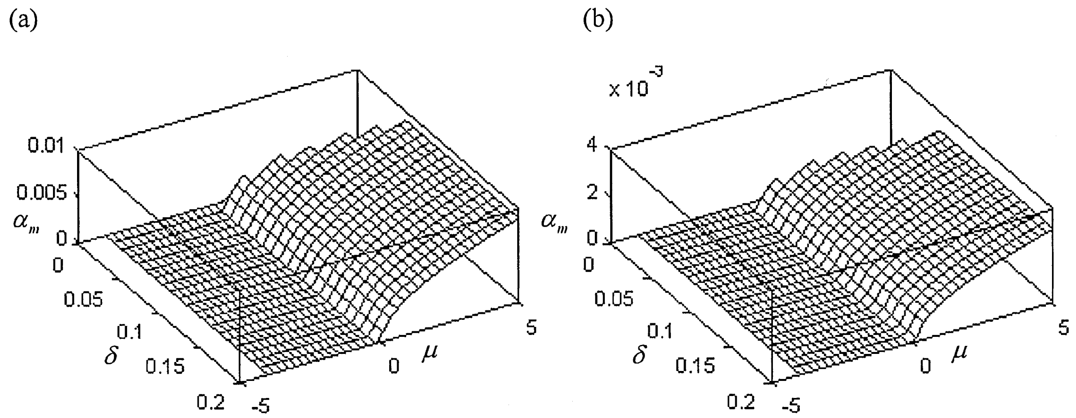

Figure 2. The response amplitude *vs.* δ and μ for fixed $\gamma = 0{\cdot}9$. (a) The first nontrivial solution. (b) The second nontrivial solution.

state response and existence boundary *vs.* the nondimensional amplitude of speed variation δ and the nondimensional the mean transport speed γ is illustrated in Figure 4.

5. Stability of trivial and nontrivial solutions

The stability of the trivial solution is treated first. Taking out the nonlinear terms from equation (25), one obtains

$$A_m' - \frac{\delta\gamma}{2}\sin(\gamma m\pi)\left[\sin(\gamma m\pi) + \mathrm{i}\cos(\gamma m\pi)\right]\bar{A}_m \mathrm{e}^{\mathrm{i}\mu T_1} = 0\ . \tag{34}$$

Suppose that the perturbed solutions of Equation (34) take the form

$$A_m = (a_r + \mathrm{i}a_i)\,\mathrm{e}^{\beta T_1 + \frac{\mathrm{i}\mu T_1}{2}}\ , \tag{35}$$

where a_r and a_i are real functions. Substituting Equation (35) in Equation (34), and separating the real and imaginary parts from the resulting equations, one has

$$\begin{aligned} &\left[2\beta - \delta\gamma\sin^2(\gamma m\pi)\right]a_r - \left[\mu + \delta\gamma\sin(\gamma m\pi)\cos(\gamma m\pi)\right]a_i = 0\ ,\\ &\left[\mu - \tfrac{\delta\gamma}{2}\delta\gamma\sin(\gamma m\pi)\cos(\gamma m\pi)\right]a_r + \left[2\beta + \delta\gamma\sin^2(\gamma m\pi)\right]a_i = 0\ . \end{aligned} \tag{36}$$

A disturbed trivial solution should be nontrivial. Hence the determinant of the coefficient matrix in equation (36) must vanish, *i.e.*,

$$4\beta^2 - \delta^2\gamma^2\sin^2(\gamma m\pi) + \mu^2 = 0\ . \tag{37}$$

β has a positive real component if

$$\delta^2\gamma^2\sin^2(\gamma m\pi) - \mu^2 > 0\ . \tag{38}$$

Hence the trivial solution of Equation (34) is unstable under this condition. The Lyapunov linearized stability theory indicates that the instability of a nonlinear system coincides with that of the corresponding linear system. Therefore the trivial solution of the two-to-one parametric resonance is unstable if

$$-\delta\gamma\,|\sin(\gamma m\pi)| < \mu < \delta\gamma\,|\sin(\gamma m\pi)|\quad . \tag{39}$$

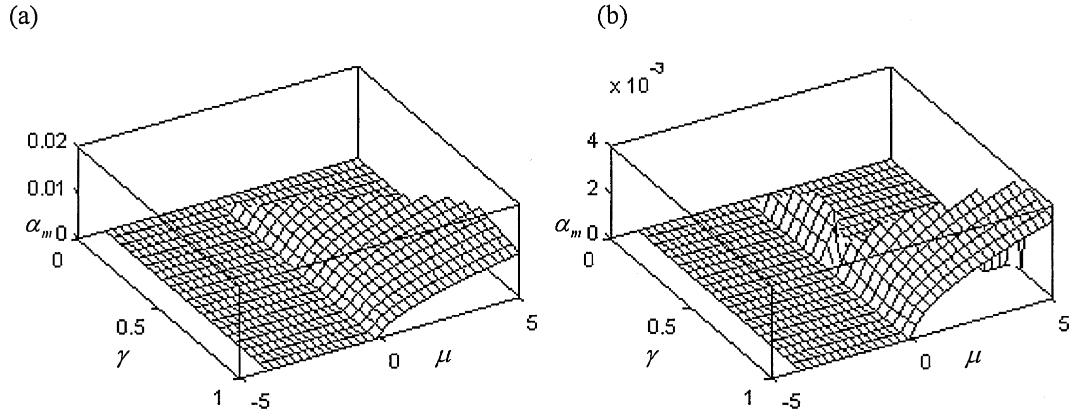

Figure 3. The response amplitude *vs.* γ and μ for fixed $\delta = 0{\cdot}1$. (a) The first nontrivial solution. (b) The second nontrivial solution.

The instability regions of the trivial solution of the two-to-one parametric resonance are respectively regions 1 in Figure 5(a) for $m = 1$ and regions 1 and 2 in Figure 5(b) for $m = 2$.

Then the stability of the nontrivial solutions is treated. Linearizing Equations (27) and (28), one obtains the linear equation

$$\begin{pmatrix} \alpha_m' \\ \theta_m' \end{pmatrix} = \mathbf{A} \begin{pmatrix} \alpha_m \\ \theta_m \end{pmatrix}, \tag{40}$$

where the matrix $\boldsymbol{A}$, the Jacobian calculated at the fixed points (α_m, θ_m), is given by

$$\mathbf{A} = -\frac{\pi^3}{2}\left(3 + 2\gamma^2 + 3\gamma^4\right) \begin{pmatrix} m^4 E_v\left(1-\gamma^2\right)\alpha_m^2 & \alpha_m\left(\dfrac{\mu}{\pi^3\left(3+2\gamma^2+3\gamma^4\right)} - \dfrac{3}{4}E_e m^3 \alpha_m^2\right) \\ 3m^3 E_e \alpha_m & m^4 E_v\left(1-\gamma^2\right)\alpha_m^2 \end{pmatrix}. \tag{41}$$

According to the Lyapunov linearized stability theory, the stability of the nontrivial solutions is decided by the nature of the eigenvalues of the matrix $\boldsymbol{A}$. If the eigenvalues have negative real parts, the steady-state solutions are stable. On the other hand, if the real part of at least one of the eigenvalues is positive, then the steady-state solution is unstable. The characteristic equation of the matrix $\boldsymbol{A}$ is

$$\lambda^2 + a_1\lambda + a_2 = 0\ , \tag{42}$$

where

$$a_1 = \pi^3 m^4 E_v\left(1-\gamma^2\right)\left(3+2\gamma^2+3\gamma^4\right)\alpha_m^2$$

$$a_2 = \tfrac{\pi^3}{4} m^3\left(3+2\gamma^2+3\gamma^4\right)\left\{\tfrac{\pi^3}{4} m^3\left(3+2\gamma^2+3\gamma^4\right)\left[4m^2\left(1-\gamma^2\right)^2 E_v^2 + 9E_e^2\right]\alpha_m^2 - 3E_e\mu\right\}\alpha_m^2. \tag{43}$$

By the use of the Routh-Hurwitz criterion, the stability conditions can be determined as

$$a_1 > 0, \tag{44}$$

$$a_2 > 0. \tag{45}$$

For subcritical transport speed, the inequality (44) holds. Substitution of Equations (31) and (43) in inequality (45) and performing complicated manipulations result in

$$-\delta\gamma\left|\sin(\gamma m\pi)\right|\sqrt{1+\frac{9E_e^2}{4m^2\left(1-\gamma^2\right)^2E_v^2}} < \mu \leq -\delta\gamma\left|\sin(\gamma m\pi)\right| \tag{46}$$

or

$$\delta\gamma\left|\sin(\gamma m\pi)\right| < \mu \leq \delta\gamma\left|\sin(\gamma m\pi)\right|\sqrt{1+\frac{9E_e^2}{4m^2\left(1-\gamma^2\right)^2E_v^2}} \tag{47}$$

for the first nontrivial solution, and

$$-\delta\gamma\left|\sin(\gamma m\pi)\right| < \mu < \delta\gamma\left|\sin(\gamma m\pi)\right| \tag{48}$$

or

$$\mu < -2mK_2(\gamma,\delta) - \delta\gamma\left|\sin(\gamma m\pi)\right|\sqrt{1+\frac{9E_e^2}{4m^2\left(1-\gamma^2\right)^2E_v^2}} \tag{49}$$

or

$$\mu > \delta\gamma\left|\sin(\gamma m\pi)\right|\sqrt{1+\frac{9E_e^2}{4m^2\left(1-\gamma^2\right)^2E_v^2}} \tag{50}$$

for the second nontrivial solution. Combining with the existence condition given by the inequalities (33), one obtains that

$$-2mK_2(\gamma,\delta) + \delta\gamma\left|\sin(\gamma m\pi)\right| < \mu < -2mK_2(\gamma,\delta) + \delta\gamma\left|\sin(\gamma m\pi)\right| \sqrt{1+\frac{9E_e^2}{4m^2\left(1-\gamma^2\right)^2E_v^2}} \tag{51}$$

are the sufficient conditions that the first nontrivial solution is stable, and the inequalities (48) are the sufficient conditions that the second nontrivial solution is stable. Inequalities (39) and (48) indicate that the stability condition of the second nontrivial solution coincides with the instability condition of the trivial solution. Inequalities (51) and (48) imply that the lower boundary of the stability region of the first nontrivial solution coincides with the upper boundary of the stability region of the second nontrivial solution. Choose $E_e = 400$ and $E_v = 10$. The stability regions of the first nontrivial solution of the two-to-one parametric resonance are plotted in Figures 6(a) and (b) as region 1 and region 2 for $m = 1, 2$, respectively.

6. Conclusions

The nonlinear transverse vibrations of the axially accelerating string have been investigated. The nonlinear partial-differential equation (6) that governs the transverse motion of the string was established based on the Newton law. The method of multiple scales has been applied directly to the partial-differential equation. Closed-form solutions (31) for the amplitude of

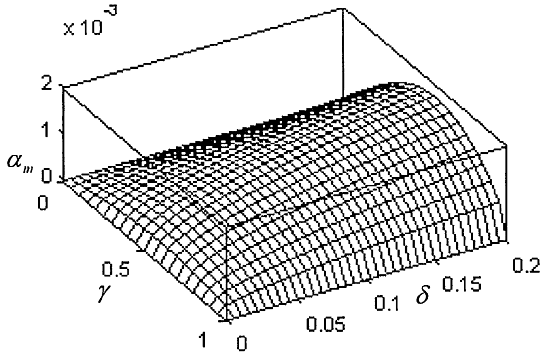

Figure 4. The first nontrivial solution at exact two-to-one parametric resonance *vs.* γ and δ.

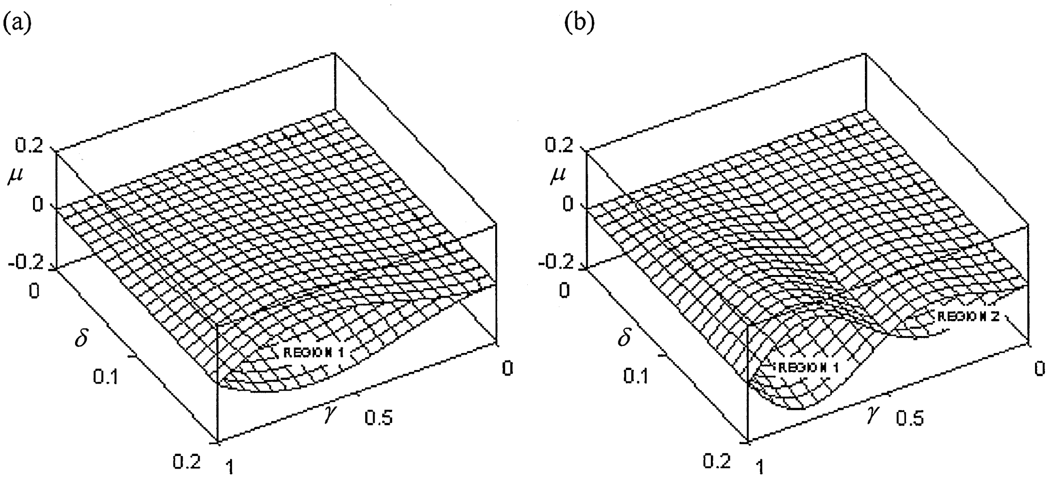

Figure 5. The instability region of the trivial solution. (a) $m = 1$. (b) $m = 2$.

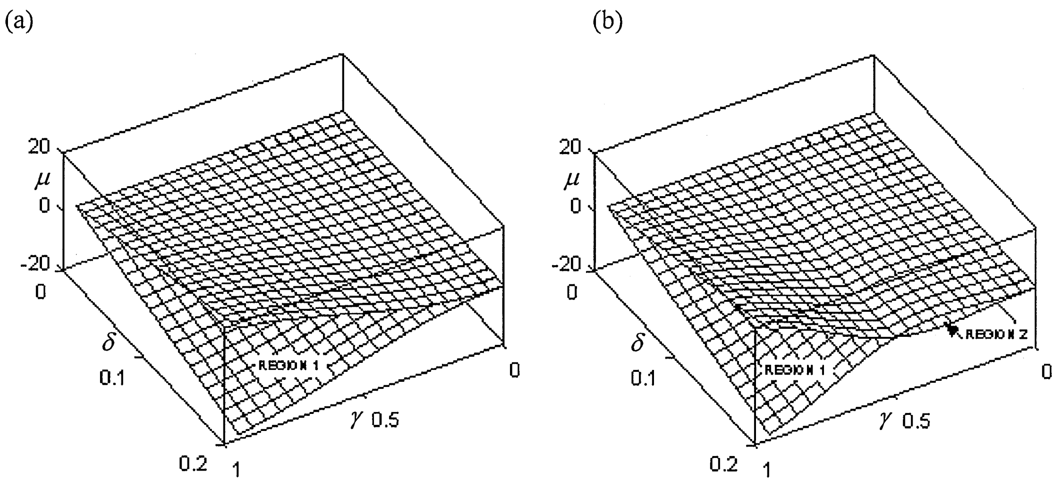

Figure 6. The stability region of the first nontrivial solution. (a) $m = 1$. (b) $m = 2$.

nontrivial response of the two-to-one parametric resonance and the corresponding existence conditions (33) are derived from the solvability conditions (25) that results in a set of differential equations (27) and (28) determining the amplitudes and the phase angles of the dynamic response. Effects of related parameters on the steady-state responses and their existence boundaries for the two-to-one parametric resonance are numerically demonstrated. By use of the Lyapunov linearized stability theory and the Routh-Hurwitz criterion, the instability conditions (39) of the trivial solution, the stability condition (41) of the first nontrivial solution and the stability condition (48) of the second nontrivial solution are presented. Effects of related parameters on these conditions have been demonstrated numerically.

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